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On certain representations of positive integers.

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by

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In this paper we investigate some properties of positive integers n, which are representable in the form $n = ux + vy$, where u and v are two positive and relatively prime integers, and x and y are non-negative integers; these integers are called representable or representable by u and v.

Without loss of generality we may suppose $u < v$.

The following properties are well known. (Confer the appendix.)

All integers $\geqq (u-1)(v-1)$ are representable by u and v. The integer $N = uv - u - v$ cannot be represented by u and v. If an integer n with $0 \leqq n \leqq N$ is representable, then $N - n$ is not, and conversely. Hence there are $\frac{1}{2}(u-1)(v-1)$ non-negative integers which cannot be represented by u and v.

In what follows P denotes the set of integers which are representable by u and v and which are $\leqq N$; Q denotes the set of non-negative integers which are not representable by u and v. Then $P \cup Q$ is the set $0,1,\ldots,N$. Further U denotes the set $1,\ldots,u-1$ and V denotes the set $1,\ldots,v-1$.

In order to deduce properties of the elements of P and Q we define for any c and any set M the set M+c as the set of all elements m+c where $m \in M$; further we define the set cM as the set of all elements cm where $m \in M$. Finally we shall denote the sum of the k^{th} powers of the elements of a set M by M^k.

We now prove two lemma's.

Lemma 1. If $q \in Q$ and $q \notin Q+u$, we have $q \in U$, and conversely.

Proof. Since $q \notin Q+u$, either q-u is representable or $q-u < 0$. If q-u is representable, so is q, which contradicts $q \in Q$. Hence $q < u$. From $q \in Q$ follows $q > 0$, so $0 < q < u$ i.e. $q \in U$.

Conversely if $q \in U$, the positive integer q is not representable so $q \in Q$. Further $q-u < 0$, so $q-u \notin Q$, hence $q \notin Q+u$.

Lemma 2. If $q \in Q+u$ and $q \notin Q$, we have $q \in vU$, and conversely.

Proof. Since $q \in Q+u$ we have $q > 0$ and since $q \notin Q$ two non-negative integers x and y exist with $q = ux + vy$. Further from $q \in Q+u$ follows $q-u \in Q$, so $q-u = u(x-1) + vy$ is not representable. Now $y \geqq 0$, so $x-1 < 0$, hence $x = 0$ and $q = vy$. Finally from $q \in Q+u$ follows $0 < q-u \leqq uv-u-v$, so $u < vy \leqq (u-1)v$. Thus $0 < y \leqq u-1$ and $q \in vU$.

Conversely since $q \in vU$ obviously $q \notin Q$ and further $q = vy$ with $0 < y \leq u-1$. The positive integer $q-u$ is not representable for otherwise non-negative integers x' and y' would exist with $q-u = vy-u = ux'+vy'$, hence $v(y-y') = u(x'+1)$. Herefrom follows $u \mid y-y'$ which is impossible since $0 < y-y' \leq y \leq u-1$. Hence $q \in Q+u$.

From lemma 1 and 2 follows the relation

$$Q \cup (vU) = (Q+u) \cup U. \tag{1}$$

We now prove also the relation

$$Q \cup (uV) = (Q+v) \cup V. \tag{2}$$

We therefore deduce two more lemma's.

Lemma 3. If $q \in Q$ and $q \notin Q+v$, we have $q \in V$ and $u \nmid q$, and conversely.

Proof. Since $q \notin Q+v$, either $q-v$ is representable or $q-v < 0$. If $q-v$ is representable, so is q, which contradicts $q \in Q$. Hence $q < v$. From $q \in Q$ follows $q > 0$, so $0 < q < v$ i.e. $q \in V$. Further since $q \in Q$ we have $u \nmid q$.

Conversely if $q \in V$ and $u \nmid q$ the integer q is not representable so $q \in Q$. Further $q-v < 0$, so $q-v \notin Q$, hence $q \notin Q+v$.

Lemma 4. If $q \in Q+v$ and $q \notin Q$, we have $q \in uW$, where W denotes the set $\left[\frac{v}{u}\right] + 1, \ldots, v-1$, and conversely.

Proof. Since $q \in Q+v$ we have $q > 0$ and since $q \notin Q$, two non-negative integers x and y exist with $q = ux+vy$. Further from $q \in Q+v$ follows $q-v \in Q$ so $q-v = ux + y(y-1)$ is not representable. Now $x \geq 0$, so $y-1 < 0$, hence $y = 0$ and $q = ux$. Finally from $q \in Q+v$ follows $0 < q-v \leq uv-u-v$, so $v < ux \leq (v-1)u$. Thus $\left[\frac{v}{u}\right] + 1 \leq x \leq v-1$ and $q \in uW$.

Conversely since $q \in uW$ obviously $q \notin Q$ and further $q = ux$ with $\left[\frac{v}{u}\right] + 1 \leq x \leq v-1$. The positive integer $q-v$ is not representable for otherwise non-negative integers x' and y' would exist with $q-v = ux-v = ux'+vy'$, hence $u(x-x') = v(y'+1)$. Herefrom follows $v \mid x-x'$ which is impossible since $0 < x-x' \leq x \leq v-1$. Hence $q \notin Q+v$.

From lemma 3 and 4 follows

$$Q \cup (uW) = (Q+v) \cup Z, \tag{3}$$

where Z denotes the set of all elements of V which are not divisible by u. If in (3) we add on both sides the set with elements $u, 2u, \ldots, \left[\frac{v}{u}\right]u$, we obtain the relation (2).

We now deduce a formula for Q^k for non-negative integers k. First we mention a few properties of the polynomials $B_h(x)$ of Bernoulli which enable us to calculate the U^k.

From

$$u^k + U^k = (U+1)^k + 1 = \sum_{h=0}^{k} \binom{k}{h} U^h + 1$$

follows

$$\sum_{h=0}^{k-1} \binom{k}{h} U^h = u^k - 1. \tag{4}$$

On the other hand we have

$$B_{k+1}(x) - B_{k+1}(x-1) = (k+1)(x-1)^k,$$

so

$$U^k = \frac{1}{k+1}(B_{k+1}(u) - B_{k+1}(1)),$$

hence, using the formula

(5) $$B_{k+1}(x) = \sum_{h=0}^{k+1} \binom{k+1}{h} x^h B_{k+1-h}$$

we get

(6) $$U^k = \frac{1}{k+1} \sum_{h=1}^{k+1} \binom{k+1}{h} (u^h-1) B_{k+1-h} = \frac{1}{k+1} \sum_{t=0}^{k} \binom{k+1}{t+1} (u^{t+1}-1) B_{k-t}.$$

We can interpret our result as follows. From the equation (4) taken for $k = 1,\dots,K$, which equation is linear in the unknowns $U^o,\dots,U^{K-1}$ these unknowns can be found and obviously are a linear compositum of the right hand members $u-1, u^2-1,\dots,u^K-1$ of the equations (4). These values of the unknowns are given by (6).

These results are used now to determine Q_k. Taking the sum of the k^{th} powers of all elements in both sides of the formula (1) we get, since $Q \cap (uV) = (Q+u) \cap U$ is empty, the relation

$$Q^k + v^k U^k = (Q+u)^k + U^k$$

hence

$$\sum_{h=0}^{k-1} \binom{k}{h} u^{k-h} Q^h = (v^k-1)U^k,$$

so

$$\sum_{h=0}^{k-1} \binom{k}{h} \frac{Q^h}{u^h} = \frac{v^k-1}{u^k} U^k.$$

Now if in the equations (4) we replace the unknowns U^h by $\frac{Q^h}{u^h}$ and the right hand sides u^k-1 by $\frac{v^k-1}{u^k} U^k$, we obtain the equations (7). Hence by the above remark the values of $\frac{Q^h}{u^h}$ must be found from (6) by the same substitution i.e.

$$\frac{Q^k}{u^k} = \frac{1}{k+1} \sum_{t=0}^{k} \binom{k+1}{t+1} \frac{v^{t+1}-1}{u^{t+1}} U^{t+1} B_{k-t},$$

and substituting in this last result for U^{t+1} its value given by (6) we get

(8) $$Q^k = \frac{1}{k+1} \sum_{t=0}^{k} \binom{k+1}{t+1} (v^{t+1}-1) u^{k-t-1} B_{k-t} \frac{1}{t+2} \sum_{s=0}^{t+1} \binom{t+2}{s+1} (u^{s+1}-1) B_{t+1-s}.$$

To reduce the last member of (8) we first calculate the expression

(9) $$\frac{1}{k+1} \sum_{t=0}^{k} \binom{k+1}{t+1} (v^{t+1}-1) u^{k-t-1} B_{k-t} \frac{1}{t+2} \sum_{s=0}^{t+1} \binom{t+2}{s+1} B_{t+1-s}.$$

Now we have from (5) with $x = 1$

$$\sum_{h=0}^{t+2} \binom{t+2}{h} B_{t+2-h} = B_{t+2}(1),$$

so

$$\sum_{h=1}^{t+2} \binom{t+2}{h} B_{t+2-h} = B_{t+2}(1) - B_{t+2}(0) = (t+2)B^{(0)}_{t+1}(0) = 0$$

since $t+1 \geqq 1$. Thus the expression $\sum_{s=0}^{t+1} \binom{t+2}{s+1} B_{t+1-s}$ vanishes and so does (9). Hence (8) reduces to

$$Q^k = \frac{1}{k+1} \sum_{t=0}^{k} \binom{k+1}{t+1}(v^{t+1}-1)u^{k-t-1}\ B_{k-t}\ \frac{1}{t+2} \sum_{s=0}^{t+1} \binom{t+2}{s+1} u^{s+1}\ B_{t+1-s} =$$

$$= \frac{1}{k+t} \sum_{t=-1}^{k}\ \sum_{s=0}^{t+1} \binom{k+1}{t+1}\binom{t+2}{s+1}\frac{1}{t+2}(v^{t+1}-1)u^{k+s-t}\ B_{k-t}\ B_{t+1-s},$$

where in the first sum the term with $t = -1$ which vanishes, has been added. Putting $k-t = i$, $t+1-s = j$ we get

$$Q^k = \frac{1}{k+1} \sum_{i=0}^{k+1}\ \sum_{j=0}^{k+1-i} \binom{k+1}{i}\binom{k-i+2}{j}\frac{B_i B_j}{k-i+2}(v^{k-i+1}-1)u^{k-j+1} =$$

(10)

$$\sum_{\substack{i,j\geqq 0\\ i+j\leqq k+1}} \frac{k!B_iB_j}{i!j!(k+2-i-j)!}\ v^{k-i+1}\ u^{k-j+1} - C,$$

where

$$C = \sum_{\substack{i,j\geqq 0\\ i+j\leqq k+1}} \frac{k!B_iB_j}{i!j!(k+2-i-j)!}\ u^{k-j+1} =$$

$$= k! \sum_{j=0}^{k+1} \frac{B_j u^{k-j+1}}{j!} \sum_{i=0}^{k+1-j} \frac{B_i}{i!(k+2-i-j)!} =$$

$$= k! \sum_{j=0}^{k+1} \frac{B_j u^{k-j+1}}{j!}\ \frac{B_{k+2-j}(1)-B_{k+2-j}}{(k+2-j)!}.$$

Here we used (5) with $x = 1$ and $k+2-j$ instead of $k+1$.

Now for $k+2-j > 1$ we have $B_{k+2-j}(1) = B_{k+2-j}$ and for $k+2-j = 1$ we have $B_{k+2-j}(1) = B_{k+2-j}+1$. So we find

$$C = k!\ \frac{B_{k+1}}{(k+1)!} = \frac{B_{k+1}}{k+1}$$

and then from (10)

$$Q^k = \sum_{\substack{i,j\geqq 0\\ i+j\leqq k+1}} \frac{k!B_iB_j}{i!j!(k+2-i-j)!}\ v^{k-i+1}\ u^{k-j+1} - \frac{B_{k+1}}{k+1}.$$

This result may symbolically be written in the form

$$Q^k = \frac{u^{k+1}v^{k+1}}{(k+1)(k+2)} \left\{(1+\frac{B}{u}+\frac{B}{v})^{k+2} - (\frac{B}{u}+\frac{B}{v})^{k+2}\right\} - \frac{B_{k+1}}{k+1},$$

where in the ordinary expansion of the $(k+2)^{th}$ powers instead of B^h has to be taken B_h.

If we take $k = 0$ we find the above formula $Q^0 = \frac{1}{2}(u-1)(v-1)$ for the number of elements of Q.

Appendix.

Above we used some results of which easily a proof is given by the following considerations.

Let as before u and v denote two integers > 1 with $(u,v) = 1$. Let $\binom{n}{u,v}$ denote the number of different ways in which the integer n can be written in the form $n = ux + vy$ with non-negative integers x and y. Then obviously

$$\frac{1}{(1-z^u)(1-z^v)} = \sum_{n=0}^{\infty} \binom{n}{n,v} z^n.$$

Since $(u,v) = 1$ the expression

$$\frac{(1-z^{uv})(1-z)}{(1-z^u)(1-z^v)}$$

is a polynomial in z of degree N+1 where $N = uv-u-v$. Hence we have

$$\frac{(1-z^{uv})(1-z)}{(1-z^u)(1-z^v)} = \sum_{n=0}^{N+1} \binom{n}{u,v} z^n - \sum_{n=0}^{N} \binom{n}{u,v} z^{n+1} =$$

$$= (1-z)\sum_{n=0}^{N} \binom{n}{u,v} z^n + \binom{N+1}{u,v} z^{N+1}.$$

Obviously the coefficient of z^{N+1} is the expansion is equal to 1, so

$$(11) \qquad \frac{1-z^{uv}}{(1-z^u)(1-z^v)} = \sum_{n=0}^{N} \binom{n}{u,v} z^n + \frac{z^{N+1}}{1-z}.$$

Replacing z by $\frac{1}{z}$ and multiplying by z^N we get

$$(12) \qquad \frac{z^{uv}-1}{(z^u-1)(z^v-1)} = \sum_{n=0}^{N} \binom{n}{u,v} z^{N-n} + \frac{1}{z-1} = \sum_{n=0}^{N} \binom{N-n}{u,v} z^n + \frac{1}{z-1}.$$

Comparing (11) and (12) we get for $n = 0,1,\dots,N$

$$\binom{n}{u,v} + \binom{N-n}{u,v} = 1.$$

Since for all n we have $\binom{n}{u,v} \geq 0$, we get for $n = 0,1,\dots,N$ the result $\binom{n}{u,v} = 0$ or 1, so all these integers n are either not representable or are representable in exactly one way. Further we get from (11)

$$\sum_{n=0}^{\infty} \binom{n}{u,v} z^n = \frac{1}{(1-z^u)(1-z^v)} = \frac{z^{N+1}}{1-z} + \frac{z^{uv}}{(1-z^u)(1-z^v)} + \sum_{n=0}^{N} \binom{n}{u,v} z^n$$

where for $n \geqq N+1$ the coefficient of z^n in the right hand side is obviously $\geqq 1$. So everey integer $n \geqq N+1$ is representable.

Corollary. If in (12) we take $z = 2$ we get

$$\frac{2^{uv}-1}{(2^u-1)(2^v-1)} - 1 = \sum_{n=0}^{N} \binom{N-n}{u,v} 2^n.$$

Now the coefficient $\binom{N-n}{u,v} = 1$ if $\binom{n}{u,v} = 0$ i.e. if n is not representable and $\binom{N-n}{u,v} = 0$ if $\binom{n}{u,v} = 1$ i.e. if n is representable. If therefore the integer $\frac{2^{uv}-1}{(2^u-1)(2^v-1)} - 1$ is written in binary scale the places of the zero's correspond with the non-representable integers and the place of the numbers 1 with the integers which are representable.